First Computation in ChiFT – Part II

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Two Finite Potential Holes

The hamiltonian reads

$$H_{2,V_0} = H_{L,V_0} + H_{R,V_0}, \tag{1}$$

where

$$H(x) = \begin{cases} 0 & \text{dla} \quad x \in \left(-\infty, -\frac{a}{2}\right) \\ -V_0 & \text{dla} \quad x \in \left(-\frac{a}{2}, \frac{a}{2}\right) \\ 0 & \text{dla} \quad x \in \left(\frac{a}{2}, \infty\right). \end{cases}$$

and

$$H_{L,V_0}(x) = H(x+a), \quad H_{R,V_0}(x) = H(x-a). \tag{2}$$

Parity-Invariance

The Hamiltonian is Parity-Invariant

$$\mathcal{P} H_{2,V_0}(x) \mathcal{P}^{-1} = H_{2,V_0}(-x). \tag{3}$$

It degenerates the Hamiltonian eigenstates.

The Ledder of States

States: Left- and Right- Handed

$$| n\rangle_0^L, \quad | n\rangle_0^R, \quad n = 0, 1, 2, \ldots \tag{4}$$

where

$$H_{L,V_0} | n\rangle_L^0 = E_n^0 | n\rangle_L^0, \quad H_{R,V_0} | n\rangle_R^0 = E_n^0 | n\rangle_R^0. \tag{5}$$

Notice the symmetry properties:

$$\mathcal{P} H_{L,V_0} \mathcal{P}^{-1} = H_{R,V_0}, \quad \mathcal{P} | n\rangle_L^0 = | n\rangle_R^0. \tag{6}$$

The total hamiltonian eigenstates are degenerate

$$| n\rangle_\pm^0 = \frac{1}{\sqrt{2}} \left(| n\rangle_L^0 \pm | n\rangle_R^0 \right), \quad \mathcal{P} | n\rangle_\pm^0 = \pm | n\rangle_\pm^0. \tag{7}$$

Consider small perturbation and apply the Stationary Perturbation Calculus for Degenerate Case.

$$H = H_{2,V_0} + \lambda V \tag{8}$$

Perturbation

We assume that the state can be written as

$$\mid n\rangle = \mid n\rangle^0 + \lambda \mid n\rangle^1 + \lambda^2 \mid n\rangle^2 + ... \tag{9}$$

and the total energy of the n-th state reads

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + ... \tag{10}$$

Degeneracy

Now the problem is that if the sates are degenerated then the standard derivation will lead to the singular solutions. The point is that the perturbation will somehow breaks the degeneracy, but in particular way. It means that every state

$$\mid n\rangle^{i} \tag{11}$$

has to be replaced by the eigenvector of (8).

$$\mid n\rangle^{0} \rightarrow \overline{\mid n\rangle^{0}} = \sum_{j=+,-} C_j^n \mid n\rangle_{\pm}^{0}. \tag{12}$$

Then the state after perturbation reads

$$\mid n\rangle = \overline{\mid n\rangle^{0}} + \lambda\overline{\mid n\rangle^{1}} + \lambda^{2}\overline{\mid n\rangle^{2}} + \ldots \tag{13}$$

Now we take all the equations together:

$$\left(H_{2,V_0} + \lambda V\right)\left(\overline{|\ n\rangle^0} + \lambda\overline{|\ n\rangle^1} + \lambda^2\overline{|\ n\rangle^2} + ...\right) = \tag{14}$$

$$\left(E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + ...\right)\left(\overline{|\ n\rangle^0} + \lambda\overline{|\ n\rangle^1} + \lambda^2\overline{|\ n\rangle^2} + ...\right) \tag{15}$$

For the ground states We have the equations:

$$H_{2,V_0}\overline{|\ 0\rangle^0} = E_0^0\overline{|\ 0\rangle^0}$$

$$H_{2,V_0}\overline{|\ 1\rangle^0} + V\overline{|\ 0\rangle^0} = E_n^0\overline{|\ 0\rangle^1} + E_0^1\overline{|\ 0\rangle^0} \tag{16}$$

$$... = ... \tag{17}$$

Take into consideration the second equation

$$ {}_j^0\langle 0 \mid H_{2,V_0}\overline{\mid 1\rangle^0} +_j^0 \langle 0 \mid V\overline{\mid 0\rangle^0} =_j^0 \langle 0 \mid E_n^0\overline{\mid 0\rangle^1} +_j^0 \langle 0 \mid E_0^1\overline{\mid 0\rangle^0} \tag{18} $$

Then we obtain

$$ \sum_{i=-,+} {}_j^0\langle 0 \mid V \mid 0\rangle_i^0 C_i^0 = E_0^1 C_j^0, \tag{19} $$

which can be re-written as it follows

$$ \sum_{i=-,+} V_{ji} C_i^0 = E_0^1 C_j^0, \tag{20} $$

It is the eigen-equation.

Solution

Consider the potential which is odd in x e.g.

$$V(x) = x. \tag{21}$$

The eigenvalue equation reads

$$\begin{vmatrix} {}^0_+\langle 0 \mid V \mid 0\rangle^0_+ - E^1_0 & {}^0_+\langle 0 \mid V \mid 0\rangle^0_- \\ {}^0_-\langle 0 \mid V \mid 0\rangle^0_+ & {}^0_-\langle 0 \mid V \mid 0\rangle^0_- - E^1_0 \end{vmatrix} = 0.$$

Now notice the properties

$${}^0_+\langle 0 \mid V \mid 0\rangle^0_+ = {}^0_+\langle 0 \mid \mathcal{P} V \mathcal{P}^{-1} \mid 0\rangle^0_+ = -{}^0_+\langle 0 \mid V \mid 0\rangle^0_+ \Rightarrow {}^0_+\langle 0 \mid V \mid 0\rangle^0_+ \tag{22}$$

Analogically

$${}^0_-\langle 0 \mid V \mid 0\rangle^0_- = 0 \tag{23}$$

Then let see

$$a \equiv {}^0_+\langle 0 \mid V \mid 0\rangle^0_- = -{}^0_+\langle 0 \mid \mathcal{P} V \mathcal{P}^{-1} \mid 0\rangle^0_- = {}^0_+\langle 0 \mid V \mid 0\rangle^0_- = {}^0_-\langle 0 \mid V \mid 0\rangle^{0\,*}_+ \tag{24}$$

After that we have

$$\begin{vmatrix} -E_0^1 & a \\ a^* & -E_0^1 \end{vmatrix} = (E_0^1)^2 - |a|^2 = 0 \quad \Rightarrow E_0^1 \pm |a|.$$

It means that the new energy of the prior-ground states read

$$E_0 = E_0^0 \pm \lambda |a|. \tag{25}$$

It means that the degeneracy of the ground state vanishes. And one of the prior ground states is chosen for the real vacuum.

Goldstone Theorem One More Time

Consider the Lagrangian depending on N scalar fields and their derivatives

$$\mathcal{L}(\phi_i, \partial_\mu \phi_j) = \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} - V(\vec{\phi}). \tag{26}$$

The lagrangian is invariant under the continues group G:

$$g \in G,: \quad \phi_i \to \phi_i + \delta\phi_i, \quad \delta\phi_i = -i\theta_a t^a_{ij} \phi_j. \tag{27}$$

$T^a = t^a_{ij}$ are hermitian, antisymmetric and purely imaginary.
Assume that the potential has the minimum at

$$\vec{\phi}_{min} = \langle \vec{\phi} \rangle, \tag{28}$$

which is invariant under subgroup H of G.

Let expand potential around this minimum, $\phi_i = \phi_i + \chi_i$

$$V(\vec{\phi}) = V(\vec{\phi}_{min}) + \underbrace{\frac{\partial V(\vec{\phi}_{min})}{\partial \phi_i}\chi_i}_{=0} + \frac{1}{2} \underbrace{\frac{\partial^2 V(\vec{\phi}_{min})}{\partial \phi_i \partial \phi_j}}_{=m_{ij}^2} \chi_i \chi_j + ... \tag{29}$$

By definition m_{ij}^2 is symmetric, and because it is around minimum, positive semidefinite, i.e.

$$\sum_{i,j} m_{ij}^2 x_i x_j \geq 0, \tag{30}$$

for every $\vec{x}$.
It means that the m_{ij}^2 matrix can be diagonalized and have nonnegative eigenvalues.

Consider

$$V(\vec{\phi}_{min}) = V(D(g)\vec{\phi}_{min}) = V(\vec{\phi}_{min}+\delta\vec{\phi}_{min}) = V(\vec{\phi}_{min})+\frac{1}{2}m^2_{ij}\delta\phi_{min,i}\delta\phi_{min,j}+... \quad (31)$$

It means that

$$m^2_{ij}\delta\phi_{min,i}\delta\phi_{min,j} = 0 \quad \Rightarrow \quad m^2_{ij}\delta\phi_{min,j} = \vec{0}. \quad (32)$$

differentiating with respect to $\delta\phi_i$ and using the symmetry of the matrix m_{ij} leads to the equation:

$$M^2\delta\vec{\phi} = \vec{0}. \quad (33)$$

Hence

$$M^2 T^a\vec{\phi}_{min} = \vec{0}. \quad (34)$$

Conclusions

The solution can be classified into two categories:

- T_a, a=1, ..., n_H is a representation of an element of the Lie algebra belonging to the subgroup H of G. Because

$$T^a\vec{\phi}_{min} = \vec{0}, \quad a = 1, ..., n_H. \tag{35}$$

- T^a, $a = n_H + 1, ..., n_G$ is not representation of any element belonging to H group. So

$$T^a\vec{\phi}_i \neq 0, \quad i = n_{H+1}, ..., n_G. \tag{36}$$

and $T^a\vec{\phi}_i$ is an eigenvector of M^2 matrix with eigenvalue 0! To each such eigenvector corresponds one Massless Goldstone Boson. $T^a\vec{\phi}_i$'s are linearly independent, let see:

$$\vec{0} = \sum_{a=n_{H+1}}^{n_G} c_a \left(T^a\vec{\phi}_{min}\right) = \left(\sum_{a=n_{H+1}}^{n_G} c_a T^a\right)\vec{\phi}_{min} \tag{37}$$

The element

$$T = \left(\sum_{a=n_{H+1}}^{n_G} c_a T^a\right) \tag{38}$$

belongs to H which is in contradiction with initial assumption.

Goldstone Theorem More General Point of View

The Noether Current is Conserved:

$$\partial_\mu J^\mu = 0, \tag{39}$$

Consider the commutator

$$0 = \int d^3x [\partial^\mu J_\mu(x,t), \phi(0)] = \partial^0 \int d^3x [J_0(x,t), \phi(0)] + \underbrace{\int d\mathbf{S} \cdot [\mathbf{J}(x,t), \phi(0)]}_{\textit{vanishing}} \tag{40}$$

Hence

$$\frac{d}{dt}[Q(x,t), \phi(0)] = 0 \tag{41}$$

The Spontaneous Symmetry Breaking Occurs if

$$\langle 0 \mid [Q(x,t), \phi(0)] \mid 0 \rangle = \eta \neq 0, \tag{42}$$

$$\begin{aligned}
\langle 0 \mid [Q(x,t),\phi(0)] \mid 0\rangle &= \sum_n \left[\langle 0 \mid Q(x,t) \mid n\rangle \langle n \mid \phi(0) \mid 0\rangle - \langle 0 \mid \phi(0) \mid n\rangle \langle n \mid Q(x,t) \mid 0\rangle\right] \\
&= (2\pi)^3 \sum_n \delta(\mathbf{p}_n) \left[e^{-iE_n t} \langle 0 \mid J(0) \mid n\rangle \langle n \mid \phi(0) \mid 0\rangle - e^{iE_n t} \underbrace{\langle 0 \mid \phi(0) \mid n\rangle \langle n \mid J(0) \mid 0\rangle}_{c_n} \right] \\
&= (2\pi)^3 \sum_n \delta(\mathbf{p}_n) \left[e^{-iE_n t} c_n^* - e^{iE_n t} c_n \right] \qquad (43) \\
&= (2\pi)^3 \sum_n \delta(\mathbf{p}_n) \left[e^{-iE_n t} c_n^* - e^{iE_n t} c_n \right] \qquad (44)
\end{aligned}$$

Define

$$c_n = |c_n| e^{i\phi_n} \qquad (45)$$

Then

$$\eta = \langle 0 \mid [Q(x,t),\phi(0)] \mid 0\rangle = -2i(2\pi)^3 \sum_n \delta(\mathbf{p}_n) |c_n| \sin(E_n t + \phi_n) \qquad (46)$$

Because η is time independent hence in order to satisfy the equality the following conditions has to be spent

$$\begin{aligned}
\delta(\mathbf{p}_n) &= 0, \quad E_n = 0 \qquad (47) \\
\langle 0 \mid J(0) \mid n\rangle &\neq 0 \quad \langle n \mid \phi(0) \mid 0\rangle \neq 0 \qquad (48)
\end{aligned}$$

Symmetry Breaking in the linear-σ model

The potential

$$V(\sigma, \vec{\pi}) = -\frac{\mu^2}{2}\left(\sigma^2 + \vec{\pi}^2\right) + \frac{\lambda}{4}\left(\sigma^2 + \vec{\pi}^2\right)^2 \tag{49}$$

is the function of the form:

$$V(x) = -\frac{\mu^2}{2}x + \frac{\lambda}{4}x^2, \quad x \geq 0. \tag{50}$$

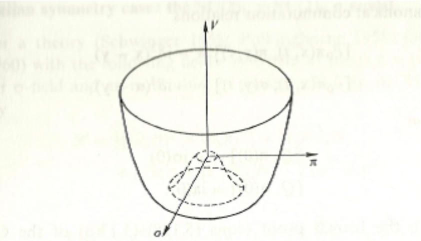


FIG. 5.3. The potential function of eqn (5.139) for $\mu^2 > 0$.

Assuming that $\lambda > 0$ then above function has the minimum, which is given by

$$0 = \frac{\partial V(x)}{\partial x} = \frac{\lambda}{2}\left(x - \frac{\mu^2}{\lambda}\right). \tag{51}$$

In the filed configuration it means that the following relation

$$\frac{\mu^2}{\lambda} = \left(\sigma^2 + \vec{\pi}^2\right). \tag{52}$$

has to be satisfied by the ground states.
Choosing one particular field configuration at minimum breaks the global symmetry, and choose one particular configuration for the vacuum. The most popular choice is:

$$\langle 0 \mid \sigma \mid 0 \rangle = v, \quad \langle 0 \mid \pi_i \mid 0 \rangle = 0,\ i = 1, 2, 3. \tag{53}$$

Useful Formulae

One can verify that after breaking the symmetry in the above way we have the following properties:

$$[\pi_i, Q^{5j}] = i\delta^{ij}\sigma, \qquad [Q^i, \pi^j] = i\epsilon^{ijk}\pi^k, \quad [Q^i, \sigma] = 0, \quad [Q^{5i}, \sigma] = i\pi^i. \tag{54}$$

Axial Symmetry Breaking

$$0 \neq \langle 0 \mid \sigma \mid 0\rangle\delta_{ij} = \langle 0 \mid \sigma\delta_{ij} \mid 0\rangle = \langle 0 \mid i[Q_{5i}, \pi_j] \mid 0\rangle \Rightarrow Q_{5i} \mid 0\rangle \neq 0. \tag{55}$$

$$0 \neq -i\delta_{ij}\langle 0 \mid \sigma \mid 0\rangle = \langle 0 \mid [Q^{5i}, \pi_j] \mid 0\rangle \tag{56}$$

$$\Rightarrow 0 \neq \langle 0 \mid Q^{5i}\pi_j \mid 0\rangle = \langle 0 \mid Q^{5i} \mid \pi_j\rangle \tag{57}$$

Notice that chosen condition are still $SU_V(2)$ invariant. Hence Q^i, $i=$1, 2, 3 remain unbroken!

One More Property

$$\langle 0 \mid [Q^{5i}, \pi_i] \mid 0 \rangle = \int d^3x \langle 0 \mid [A_0^{5i}(x), \pi_i(0)] \mid 0 \rangle \quad (58)$$

$$= \sum_n (2\pi)^3 \delta(p_n) \left(\langle 0 \mid A_0^{5i}(0) \mid n \rangle \langle n \mid \pi_i(0) \mid 0 \rangle e^{-iE_n t} - \langle 0 \mid \pi_i(0) \mid n \rangle \langle n \mid A_0^{5i}(0) \mid 0 \rangle e^{+iE_n t} \right)$$

$$\Rightarrow \quad 0 \neq \langle 0 \mid \pi_i \mid \pi \rangle, \quad 0 \neq \langle 0 \mid A_0^i \mid \pi \rangle!!! \quad \underbrace{\Rightarrow}_{from-Lorentz-covariance} \quad 0 \neq \langle 0 \mid A_\mu^i \mid \pi \rangle \quad (59)$$

$$(60)$$

The lowest hadron state $\mid n = \pi \rangle$!

Symmetry Breaking Pattern

$$SU_V(2) \times SU_A(2) \to SU_V(2). \tag{61}$$

Come Back to the Lagrangian

We expect that pions remain as the massless fields, while the sigma becomes the massive particle. This mechanism generates also the nucleon masses.

$$\sigma \to \sigma = \sigma' - v, \quad \pi \to \pi. \tag{62}$$

Then the nucleon mass is obtained

$$g\overline{N}\left(\sigma' + i\gamma_5 \vec{\tau}\cdot\vec{\pi}\right)N \to g\overline{N}\left(\sigma' - v + i\gamma_5 \vec{\tau}\cdot\vec{\pi}\right)N \tag{63}$$

hence

$$gv = M \tag{64}$$

Now look at the potential

$$V(\sigma', \vec{\pi}) \quad \to \quad \underbrace{\frac{\mu^2}{2}\sigma'^2}_{\textit{gives mass to } \sigma} \quad - \frac{\mu^2}{4}v^2 + \lambda v \sigma'(\sigma'^2 + \vec{\pi}^2) + \frac{\lambda}{4}\left(\sigma'^2 + \vec{\pi}^2\right)^2 \tag{65}$$

The lagrangian after symmetry breaking reads

$$\mathcal{L}_{broken}(x) = \frac{1}{2}\left[\partial_\mu \sigma' \partial^\mu \sigma' - \mu^2 {\sigma'}^2\right] + \frac{1}{2}\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \overline{N}\left[i\gamma^\mu \partial_\mu - gv\right] N$$
$$- g\overline{N}\left(\sigma' + i\gamma_5 \vec{\tau} \cdot \vec{\pi}\right) N - \lambda v \sigma' ({\sigma'}^2 + \vec{\pi}^2) - \frac{\lambda}{4}\left({\sigma'}^2 + \vec{\pi}^2\right)^2 \quad (66)$$

Different Choice of the Vacuum

Notice that one can propose other choice of the vacuum for instance $\langle 0 \mid \pi^2 \mid 0 \rangle \neq 0$ and zero for rest of fields. Does the symmetry breaking pattern will be different then the previous one?

Notice first that

$$0 \neq -i\langle 0 \mid \pi^2 \mid 0 \rangle = \langle 0 \mid [Q^1, \pi^1] \mid 0 \rangle \quad \Rightarrow \quad 0 \neq \langle 0 \mid Q^1 \mid 0 \rangle. \tag{67}$$

It means that Q^1 can not annihilate the vacuum. Analogically

$$0 \neq i\langle 0 \mid \pi^2 \mid 0 \rangle = \langle 0 \mid [Q^3, \pi^1] \mid 0 \rangle \quad \Rightarrow \quad 0 \neq \langle 0 \mid Q^3 \mid 0 \rangle. \tag{68}$$

It means that Q^3 can not annihilate the vacuum. Analogically

$$0 \neq i\langle 0 \mid \pi^2 \mid 0 \rangle = \langle 0 \mid [Q^{5\,2}, \sigma] \mid 0 \rangle \quad \Rightarrow \quad 0 \neq \langle 0 \mid Q^{5\,2} \mid 0 \rangle. \tag{69}$$

It means that $Q^{5\,2}$ can not annihilate the vacuum.

To every generator one can assign the Goldstone boson. One pseudoscalar (axial broken generator) and two scalar (vector broken generator) particles.
We left with Q^2, $Q^{5\,1}$ and $Q^{5\,3}$ generators. Notice that:

$$[Q^2, Q^{5\,1}] = i\epsilon_{213} Q^{5\,3}, \quad [Q^2, Q^{5\,3}] = i\epsilon_{231} Q^{5\,1}, \quad [Q^{5\,1}, Q^{5\,3}] = i\epsilon_{132} Q^2. \tag{70}$$

It means that we left with the $SU(2)$ symmetry group, but it is not vector group anymore.
Let redefine the fields

$$\pi^{2'} = \pi^2 - v \tag{71}$$

The Nucleon-meson part of Lagrangian is

$$g\overline{N}\left(\sigma + i\gamma_5 \vec{\tau}\cdot\vec{\pi}\right) N \rightarrow g\overline{N}\left(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi} - iv\gamma_5\tau_2\right) N \tag{72}$$

The mass terms reads

$$-ivg\left[\overline{N}_L\tau_2 N_R - \overline{N}_R\tau_2 N_L\right] \tag{73}$$

Chiral Transformation

One expects the mass term:

$$-\left(\overline{N'}_L N'_R + \overline{N'}_R N'_L\right), \quad (74)$$

Consider the chiral transformation:

$$-(\overline{N}_L L^\dagger i\tau_2 R N_R - \overline{N}_R R^\dagger i\tau_2 L N_L) \quad (75)$$

Then we get two equations:

$$L^\dagger i\tau_2 R = 1, \quad R^\dagger i\tau_2 L = -1. \quad (76)$$

One can guess that τ^2 operator (belongs to the remaining non-broken group) can be a good candidate for the desire operator. Let propose

$$R = 1, \quad L = \exp\left(\frac{i\theta\tau_2}{2}\right) = \cos\frac{\theta}{2} + i\tau_2 \sin\frac{\theta}{2} \quad (77)$$

Then for the $\theta = \pi$ we get the solution:

$$L = i\tau_2 \quad (78)$$

Then we generate the analogical as before the mass term. Notice that the meson fields transforms as follows:

$$\Sigma = L^\dagger \Sigma' R. \quad (79)$$

However it does not affect the structure of the Lagrangian, because the meson part is always traced!

State Transformation

Notice that

$$\Sigma' = R^\dagger \Sigma L = -i\tau_2\sigma + \tau_3{\pi'}^1\tau_3 - {\pi'}^2 - \tau_1{\pi'}^3\tau_1 \tag{80}$$

Which means that

$$\sigma \rightarrow \pi^2 \tag{81}$$
$$\pi^1 \rightarrow \pi^3 \tag{82}$$
$$\pi^2 \rightarrow \pi^2 \tag{83}$$
$$\pi^3 \rightarrow -\pi^1 \tag{84}$$

Let Pions Be Massive

Consider the potential

$$V(x) = -\frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 + c\sigma. \quad (85)$$

The conditions for the minimum of the potential reads

$$0 = \frac{\partial V}{\partial \sigma} = -\mu^2\sigma + \lambda\sigma(\sigma^2 + \vec{\pi}^2) + c \quad (86)$$

$$0 = \frac{\partial V}{\partial \pi^i} = \pi^i\left(-\mu^2 + \lambda(\sigma^2 + \vec{\pi}^2)\right) \quad (87)$$

If we choose for the minimum

$$-\mu^2 + \lambda(\sigma^2 + \vec{\pi}^2) = 0$$

then we obtain that $c = 0$ to satisfy the first equation. It means that $\pi^i = 0$, then we get only one possible solution

$$0 = -\mu^2\sigma + \lambda\sigma^3 + c \quad (88)$$

the vacume is then

$$0 = -\mu^2 v + \lambda v^3 + c. \quad (89)$$

at least one root in above expression.

Let $\sigma' = \sigma - v$ and look once again at potential, then

$$V(\sigma', \vec{\pi}) = -\sigma'^2 \left(\frac{\mu^2}{2} - \frac{3\lambda}{2} v^2 \right) - c\sigma' - \frac{c}{2v}\vec{\pi}^2 + \lambda v \sigma' (\sigma'^2 + \vec{\pi}^2) + \frac{\lambda}{4} \left(\sigma'^2 + \vec{\pi}^2 \right)^2 \quad (90)$$

we get the mesons masses

$$m_\sigma^2 = -\mu^2 + 3\lambda v^2 = 2\mu^2 - 3\frac{c}{v} \quad (91)$$

$$m_\pi^2 = -\frac{c}{v} \quad (92)$$

Notice that if one assumes that vacuum expectation $v > 0$ then for $c < 0$ pions and sigma have nonzero masses, assuming that exists c...

$$c\sigma = \frac{1}{2} Tr \left[\Sigma + \Sigma^{\dagger}\right] \tag{93}$$

The full lagrangian has the form

$$\begin{aligned} \mathcal{L}(x) &= \frac{1}{4}\mathrm{Tr}\left[\partial_\mu \Sigma \partial^\mu \Sigma^{\dagger}\right] + \overline{N}_L i\gamma^\mu \partial_\mu N_L + \overline{N}_R i\gamma^\mu \partial_\mu N_R - g\overline{N}_L \Sigma N_R - g\overline{N}_R \Sigma^{\dagger} N_L \\ &+ \frac{\mu^2}{4}\mathrm{Tr}\left(\Sigma\Sigma^{\dagger}\right) - \frac{\lambda}{16}\mathrm{Tr}\left(\Sigma\Sigma^{\dagger}\right)^2 + \frac{c}{2}\mathrm{Tr}\left[\Sigma + \Sigma^{\dagger}\right]. \end{aligned} \tag{94}$$

It is interesting to see how transforms (under $SU_V(2)$) the new, pion mass generating term.

$$\begin{aligned}
\mathrm{Tr}\left[\Sigma' + \Sigma'^{\dagger}\right] &= \mathrm{Tr}\left[V\Sigma V^{\dagger} + V^{\dagger}\Sigma V\right] && (95) \\
&= \mathrm{Tr}\left[V\Sigma V^{\dagger}\right] + \mathrm{Tr}\left[V^{\dagger}\Sigma V\right] && (96) \\
&= \mathrm{Tr}\left[V^{\dagger} V\Sigma\right] + \mathrm{Tr}\left[VV^{\dagger}\Sigma\right] = \mathrm{Tr}\left[\Sigma + \Sigma^{\dagger}\right] && (97)
\end{aligned}$$

We have obtained spectacular property, the model with the mass term (93) is still invaraiant under $SU_V(2)$ transformations! It is not the case for axial transformations.

What about divergence of the axial current

Lagrangian Once Again

$$\mathcal{L}(x) = \frac{1}{2}\left[\partial_\mu\sigma'\partial^\mu\sigma' - m_\sigma^2{\sigma'}^2 + \partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - m_\pi^2\vec{\pi}\cdot\vec{\pi}\right] + \overline{N}\left[i\gamma^\mu\partial_\mu - gv\right]N$$
$$-g\overline{N}\left(\sigma' + i\gamma_5\vec{\tau}\cdot\vec{\pi}\right)N \quad (98)$$
$$+c\sigma' - \lambda v\sigma'({\sigma'}^2 + \vec{\pi}^2) - \frac{\lambda}{4}\left({\sigma'}^2 + \vec{\pi}^2\right)^2. \quad (99)$$

E-L equations for massless pion case:

$$0 = \partial_\mu\overline{N}i\gamma^\mu + g\overline{N}(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi}) \quad (100)$$
$$0 = -i\gamma^\mu\partial_\mu N + g(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})N \quad (101)$$
$$0 = \partial_\mu\partial^\mu\sigma + g\overline{N}N$$
$$0 = \partial_\mu\partial^\mu\vec{\pi} + g\overline{N}(i\gamma_5\vec{\tau})N \quad (102)$$

For the massless pion case on can show that the divergence of the axial current:

$$\vec{A}^{\mu} \quad = \quad -\overline{N}\gamma^{\mu}\frac{\vec{\tau}}{2}\gamma_5 N + \partial_{\mu}\sigma\vec{\pi} - \sigma\partial_{\mu}\vec{\pi} \tag{103}$$

is vanishing

$$\begin{aligned}
\partial_{\mu}\vec{A}^{\mu} &= -\partial_{\mu}\overline{N}\gamma^{\mu}\frac{\vec{\tau}}{2}\gamma_5 N - \overline{N}\gamma^{\mu}\frac{\vec{\tau}}{2}\gamma_5\partial_{\mu}N + \partial^{\mu}\partial_{\mu}\sigma\vec{\pi} - \sigma\partial^{\mu}\partial_{\mu}\vec{\pi} \\
&= -ig\overline{N}(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\frac{\vec{\tau}}{2}\gamma_5 N - i\overline{N}\frac{\vec{\tau}}{2}\gamma_5 g(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})N - g\overline{N}N\vec{\pi} + \sigma g\overline{N}(i\gamma_5\vec{\tau})N \\
&= -ig\overline{N}(\sigma\gamma_5\vec{\tau})N + \overline{N}\left(\vec{\tau}\cdot\vec{\pi}\frac{\vec{\tau}}{2} + \frac{\vec{\tau}}{2}\vec{\tau}\cdot\vec{\pi}\right)N + g\overline{N}N\vec{\pi} + \sigma g\overline{N}(i\gamma_5\vec{\tau})N \\
&= 0
\end{aligned} \tag{104}$$

Massive Pion Mass Case

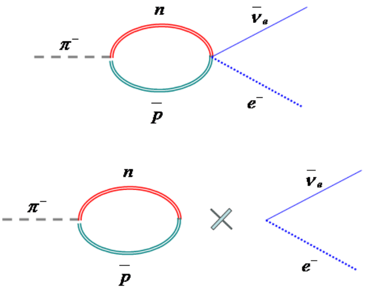
If $c\sigma$ is added to the potential. The the E-L equation for σ reads

$$0 = \partial_\mu \partial^\mu \sigma + g\overline{N}N + \mu^2 \sigma - \lambda\sigma(\sigma^2 + \vec{\pi}^2) + c \tag{105}$$

It means that

$$\partial_\mu \vec{A}^\mu = c\vec{\pi} = -v m_\pi^2 \vec{\pi}. \tag{106}$$

It shows that the axial current diverges!!!



$$T = \underbrace{\frac{f_\pi}{m_\pi}}_{pion-hadron-structure} \underbrace{p^\lambda}_{to-get-Lorenz-Invariance} \times \overline{u}_\nu \gamma_\lambda (1-\gamma_5) u_e$$

Notice that the hadronic matrix element (if axial current is universal) reads

$$\frac{f_\pi}{m_\pi} p^\lambda = \langle 0 \mid A^\lambda \mid \pi \rangle \tag{107}$$

The relation between vacuum expectation value and pion coupling constant

The constant c can be related to the pion mass and the pion decay constant and f_π. For the $\pi \to \mu\nu_\mu$ decay, the amplitude is proportional to the axial current matrix element, which defines f_π by

$$\langle 0 \mid A^a_\mu(0) \mid \pi^a(p)\rangle = i\delta^{ab} f_\pi p_\mu. \tag{108}$$

From the other hand

$$\langle 0 \mid \partial^\mu A^a_\mu(0) \mid \pi^a(p)\rangle = \delta^{ab} f_\pi m_\pi^2 = -c\langle 0 \mid \pi^a(0) \mid \pi^b(p)\rangle \tag{109}$$

Hence

$$f_\pi m_\pi^2 = -c. \tag{110}$$

Then the divergence of the axial current shows the PCAC form

$$\partial^\mu A^a_\mu = f_\pi m_\pi^2 \pi^a. \tag{111}$$

We see that

$$f_\pi = v. \tag{112}$$

It is interesting to observe that in practise the ground state is given by the condition $\sigma^2 + \vec{\pi}^2 = v^2$
in other words $\sigma^2 + \vec{\pi}^2 = f_\pi^2$.

Next Lectures

- Getting the Goldberger-Treiman Formula
- Does Linear Sigma Model is really low-energy realization of the QCD?
- What are the relations between linear, nonlinear and other effective approaches?
- QCD a theory without explicit symmetry breaking terms...