Semi-rigorous statistical inference: fitting multiplicities in Au-Au collisions at $\sqrt{s_{NN}} = 62.4$ and 200 GeV

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Seminarium ZFN
20 stycznia 2014

Sources

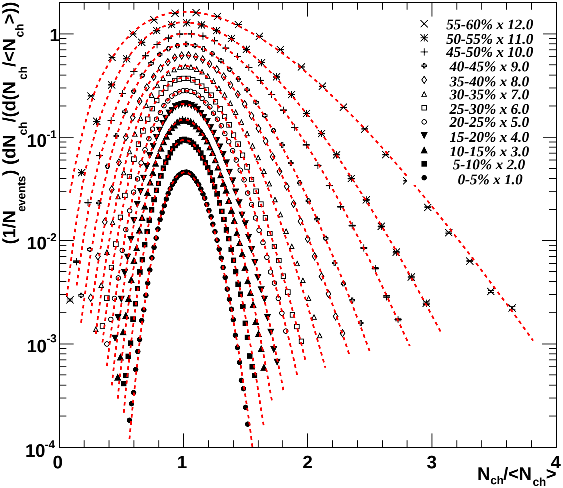
DP, Phys. Rev. C **88**, 034910 (2013)

Data:

PHENIX Collaboration, Phys. Rev. C **78**, 044902 (2008)

Multiplicity: Au-Au at $\sqrt{s_{NN}} = 200$ GeV

$|\, \eta \,| < 0.26$



Multiplicity: Au-Au at $\sqrt{s_{NN}} = 62.4$ GeV

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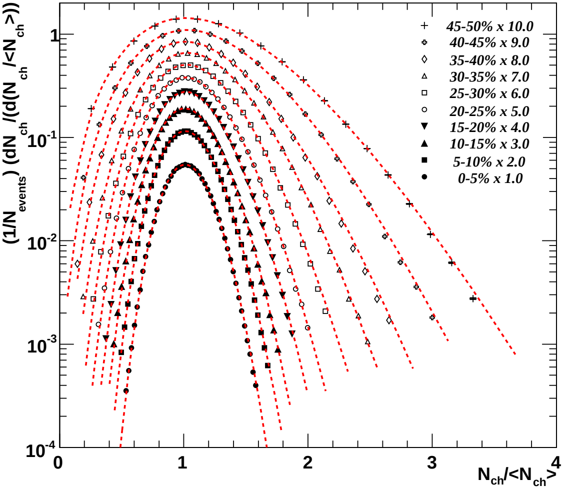


Negative binomial distribution

$$P(n;p,k) = \frac{k(k+1)(k+2)...(k+n-1)}{n!}(1-p)^n p^k$$

$0 \leqslant p \leqslant 1$, k is a positive real number

$n = 0, 1, 2, ...$ - the number of charged particles in an event

$$(k,p) \longrightarrow \left(k, \bar{n} = \frac{k(1-p)}{p}\right)$$

$\bar{n}$ - expectation value of n

The least-squares method

$$\chi^2_{LS}(\vec{Y};\vec{\theta}) = \sum_{i=1}^{N} \frac{[Y_i - \Lambda_i(\vec{\theta})]^2}{err_i^2}$$

$\vec{Y} = (Y_1, Y_2, ..., Y_N)$ - vector of data

err_i - uncertainty of the ith measurement

$\vec{\theta} = (\theta_1, \theta_2, ..., \theta_m)$ - parameters of the model

$$\chi^2_{LS,min}(\vec{Y}) = \chi^2_{LS}(\vec{Y};\vec{\hat{\theta}})$$

$\vec{\hat{\theta}}$ - LS estimators of parameters $\vec{\theta}$

What does $\chi^2_{LS,min}$ mean really?

If

- $(Y_1, Y_2, ..., Y_N)$ are independent Gaussian random variables with known variances σ_i^2,
- errors $err_i = \sigma_i$,
- the hypothesis $\Lambda_i(\theta_1, ..., \theta_m)$ is linear in the parameters θ_j ,
- the hypothesis is correct,

then the test statistic $\chi^2_{LS,min}$ is distributed according to a χ^2 distribution with $n_d = N - m$ degrees of freedom.

If the hypothesis is nonlinear, then the conclusion is valid in the limit $N \longrightarrow \infty$.

χ^2 distribution

$0 \leqslant t \leqslant +\infty$,

$n = 1, 2, ...$ - the number of degrees of freedom

$$f(t;n) = \frac{1}{2^{n/2}\Gamma(n/2)} t^{n/2-1} \cdot e^{-t/2}$$

$$E[t] = n, \qquad V[t] = 2n$$

Inference from the value of $\chi^2_{LS,min}$

$$E[\chi^2_{LS,min}] = n_d, \qquad V[\chi^2_{LS,min}] = 2n_d$$

$$\sigma[\chi^2_{LS,min}] = \sqrt{2n_d}$$

$$\boxed{\frac{\chi^2_{LS,min}}{n_d} \sim 1}$$

$$\frac{\chi^2_{LS,min}}{n_d} = 1 \pm \sqrt{\frac{2}{n_d}}$$

Goodness-of-fit: p-value

The probability of obtaining the value of the test statistic equal to or greater then the value just obtained for the present data set (*i.e.* χ^2_{min}), when repeating the whole experiment many times (repeating measurement of $\vec{Y}$):

$$p = P(\chi^2 \geqslant \chi^2_{min}) = \int_{\chi^2_{min}}^{\infty} g(t)dt \; ,$$

$g(t)$ - probability density function of χ^2_{min}, NOT KNOWN USUALLY

$\chi^2_{min}(\vec{Y})$ - statistic because a function of multidimensional random variable $\vec{Y}$

Inference from p-value: user guide

- Assume the significance level α in advance.

  ($\alpha = 0.1\%$, here)

- If $p < \alpha$, a hypothesis should be rejected ("bad fit").

- If $p \geqslant \alpha$, a hypothesis can not be rejected ("good fit").

The LS test statistic for a PHENIX histogram

$$\chi^2_{LS}(\vec{\mathbf{n}}; \bar{n}, k) = \sum_{i=1}^{m} \frac{(n_i - \nu_i(\bar{n}, k))^2}{err_i^2}$$

$\vec{\mathbf{n}} = (n_1, n_2, ..., n_m)$ - vector of data (entries)

err_i - uncertainty of the ith measurement

$\nu_i = N \cdot P(i-1; \bar{n}, k)$ - expected number of entries

$N = \sum n_i$ - total number of events

$$\chi^2_{min}(\vec{\mathbf{n}}) = \chi^2_{LS}(\vec{\mathbf{n}}; \hat{\bar{n}}, \hat{k})$$

$\hat{\bar{n}}, \hat{k}$ - estimators of parameters $\bar{n}$ and k

Errors on PHENIX multiplicity measurements

$$err_i^2 = \sigma_{i,stat}^2 + \sigma_{i,syst}^2$$

$$\sigma_{i,stat} = \sqrt{n_i}\ , \qquad \sigma_{i,syst} = 3 \cdot \sigma_{i,stat} = 3 \cdot \sqrt{n_i}$$

$$\boxed{\chi^2_{PHEN}(\vec{\mathbf{n}}; \bar{n}, k) = \frac{1}{10} \cdot \sum_{i=1}^{m} \frac{(n_i - \nu_i(\bar{n}, k))^2}{n_i} = \frac{1}{10} \cdot \chi^2_N(\vec{\mathbf{n}}; \bar{n}, k)}$$

χ^2_N - Neyman's χ^2 test statistic, asymptotically χ^2 distributed !
Jerzy Spława-Neyman (1894-1981)

$\Longrightarrow$ PHENIX χ^2 function is NOT χ^2 distributed !

Distribution of PHENIX test statistic

the distribution $g(t)$ of a function $t(z)$ of a random variable z with the known p.d.f. $f(z)$:

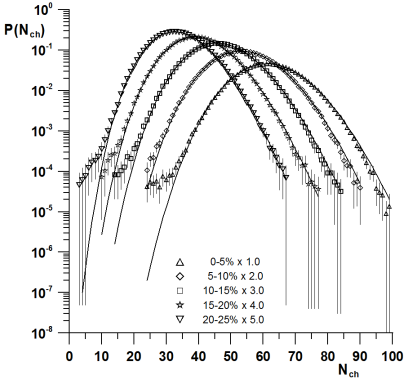
$$g(t) = f(z(t)) \left| \frac{dz}{dt} \right|$$

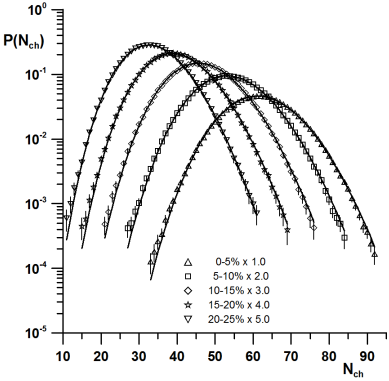
$$g(t; n_{dof}) = 10 f(10t; n_{dof})$$

p-value of PHENIX test statistic:

$$\boxed{p = \int_{10 \cdot \chi^2_{PHEN,min}}^{\infty} f(t; n_{dof}) dt}$$

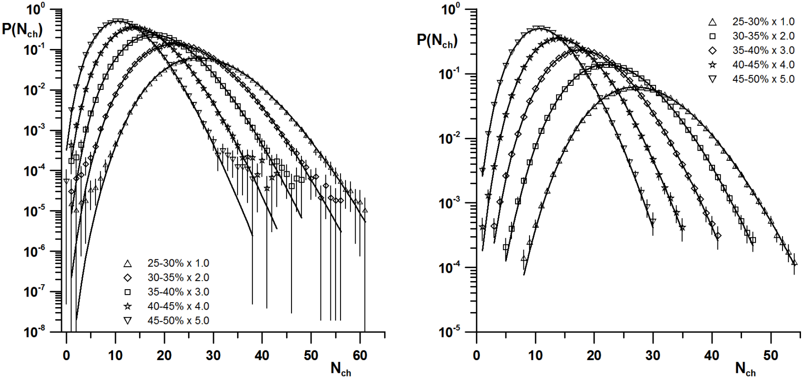
Distributions: Au-Au at $\sqrt{s_{NN}} = 200$ GeV





Rysunek: Uncorrected multiplicity distributions for bins with $n_i > 5$ (left) and $n_i > 60$ (right).

Distributions: Au-Au at $\sqrt{s_{NN}} = 200$ GeV



Rysunek: Uncorrected multiplicity distributions for bins with $n_i > 5$ (left) and $n_i > 60$ (right).

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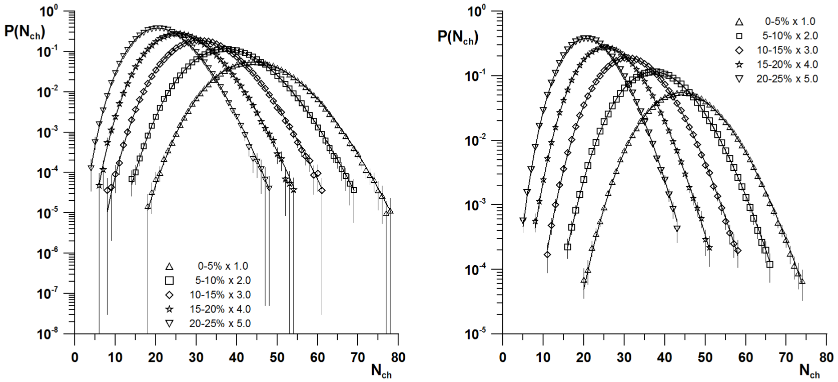
Results: Au-Au at $\sqrt{s_{NN}} = 200$ GeV

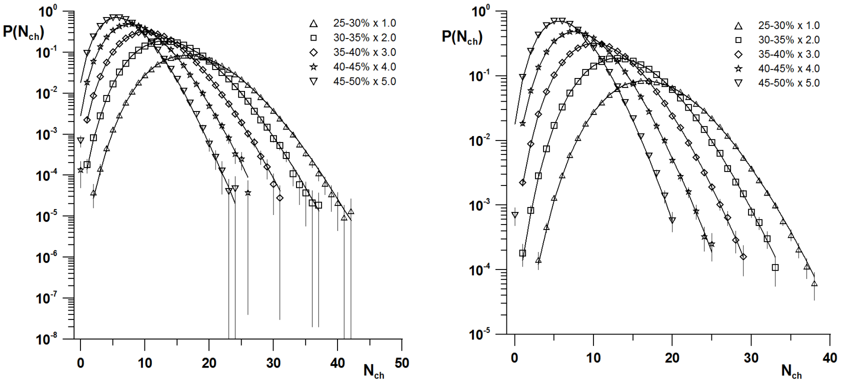
Centr. %	N	$\hat{k}$	$\hat{\bar{n}}$	χ^2_{PHEN}/n_d	p-value %
0-5	652579	289.0± 2.9	61.86± 0.01	0.57	0
5-10	657571	168.1± 1.2	53.91± 0.01	0.61	0
10-15	658258	116.4± 0.7	46.50± 0.01	0.53	0
15-20	659302	86.9± 0.5	39.72± 0.01	0.43	0
20-25	658461	69.1± 0.4	33.56± 0.01	0.34	0
25-30	659337	57.9± 0.3	28.0± 0.01	0.28	$6.7 \cdot 10^{-8}$
30-35	659021	48.3± 0.3	23.02± 0.01	0.16	0.76
35-40	660937	41.3± 0.2	18.64± 0.01	0.19	0.12
40-45	661422	34.6± 0.2	14.84± 0.01	0.21	0.015
45-50	661577	27.9± 0.2	11.56± 0.005	0.23	0.011
50-55	661877	21.9± 0.1	8.81± 0.004	0.30	$7.8 \cdot 10^{-5}$

Distributions: Au-Au at $\sqrt{s_{NN}} = 62.4$ GeV



Rysunek: Uncorrected multiplicity distributions for bins with $n_i > 5$ (left) and $n_i > 40$ (right).

Distributions: Au-Au at $\sqrt{s_{NN}} = 62.4$ GeV



Rysunek: Uncorrected multiplicity distributions for bins with $n_i > 5$ (left) and $n_i > 60$ (right).

Results: Au-Au at $\sqrt{s_{NN}} = 62.4$ GeV

Centr. %	N	$\hat{k}$	$\hat{\bar{n}}$	χ^2_{PHEN}/n_d	p-value %
0-5	607075	227.9 ± 2.5	44.67 ± 0.01	0.19	$5.6 \cdot 10^{-3}$
5-10	752263	143.9 ± 1.1	37.96 ± 0.01	0.12	14.4
10-15	752739	116.2 ± 0.9	31.53 ± 0.01	0.13	7.0
15-20	752492	88.5 ± 0.6	26.07 ± 0.01	0.11	30.9
20-25	752182	69.2 ± 0.5	21.35 ± 0.01	0.22	$2.4 \cdot 10^{-3}$
25-30	752095	53.6 ± 0.4	17.30 ± 0.01	0.23	$1.8 \cdot 10^{-3}$
30-35	751324	40.3 ± 0.3	13.84 ± 0.005	0.26	$4.3 \cdot 10^{-4}$
35-40	751639	31.8 ± 0.2	10.89 ± 0.004	0.15	3.5
40-45	750852	25.2 ± 0.2	8.42 ± 0.004	0.22	0.062
45-50	751348	22.0 ± 0.2	6.41 ± 0.003	343	0

Conclusions

1. Caution is necessary, when one infers about quality of a fit when the distribution of a test statistic is not known. Then inference from the condition $\chi^2/n_d \sim 1$ could be confused.
2. Adding statistical and systematic errors in quadrature could change properties of the LS test statistic entirely.
3. As far as PHENIX Au-Au data are concerned, only for 6 from 21 cases of collision energy and centrality the NBD hypothesis can not be rejected.